The Parametric Line Equation: Fundamental Analysis of 3D Objects

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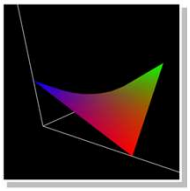

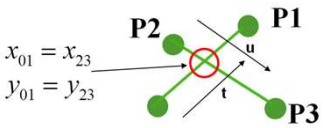

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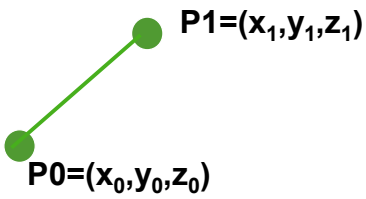
The Line Equation: Why "y=mx+b" Isn't Good Enough for Graphics and Simulation Applications



1. You cannot represent vertical lines (m = ∞)
2. You can only represent infinite lines, not finite line segments
3. You can only represent 2D lines, not 3D

Note: "z = mx + ny + b" is a *plane*, not a 3D line

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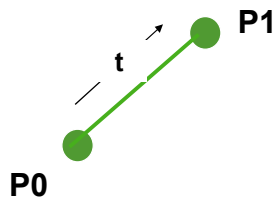
Parametric Line Equation: The "Shooting" Form

3



$$x = X_0 + t(X_1 - X_0)$$

$$y = Y_0 + t(Y_1 - Y_0)$$

$$z = Z_0 + t(Z_1 - Z_0)$$

$$0. \le t \le 1.$$

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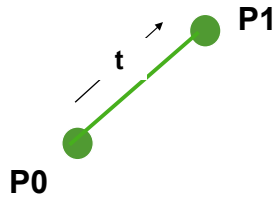
3

Parametric Line Equation: The "Blending" Form

4



$$x = (1-t)X_0 + tX_1$$

$$y = (1-t)Y_0 + tY_1$$

$$z = (1-t)Z_0 + tZ_1$$

$$0. \le t \le 1.$$

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The Shooting and Blending forms are the same equation, just re-arranged.

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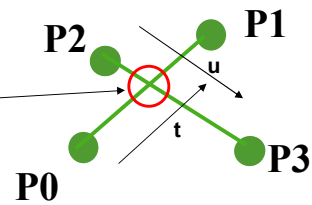
4

Example Use: Intersection of two 2D Line *Segments*

$$x_{01} = x_{23}$$
$$y_{01} = y_{23}$$



$$x_{01} = X_0 + t(X_1 - X_0) \qquad x_{23} = X_2 + u(X_3 - X_2)$$
$$y_{01} = Y_0 + t(Y_1 - Y_0) \qquad y_{23} = Y_2 + u(Y_3 - Y_2)$$

Since, at the point of intersection, $x_{01} = x_{23}$ and $y_{01} = y_{23}$

we have 2 equations in 2 unknowns (t and u)

Solve for t* and u*. If they are not each between 0. and 1., then the infinite lines intersect, but not the finite line segments. Plug t* and u* back into the equations above to find (x*,y*), the point of intersection.

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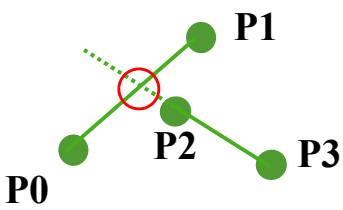
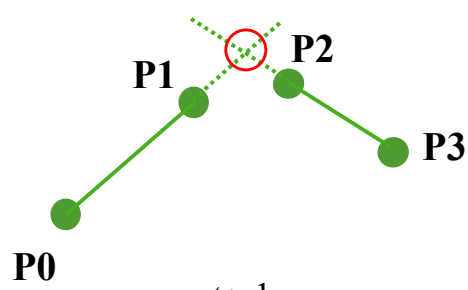
What if t* or u* is < 0. or > 1. ?



$0. \le t \le 1.$

$u < 0.$



$t > 1.$

$u < 0.$

It means you have a situation like one of these. The infinite lines intersect, but not the finite line segments.

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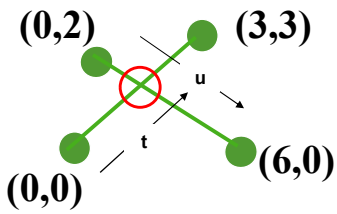
An Example with Numbers

7



$$x_{01} = x_0 + t(x_1 - x_0)$$
$$y_{01} = y_0 + t(y_1 - y_0)$$

$$x_{23} = x_2 + t(x_3 - x_2)$$
$$y_{23} = y_2 + t(y_3 - y_2)$$

$$x_{01} = 0 + t(3-0)$$
$$y_{01} = 0 + t(3-0)$$

$$x_{23} = 0 + u(6-0)$$
$$y_{23} = 2 + u(0-2)$$

$$3t = 6u$$
$$3t = 2 - 2u$$

$$t^* = \frac{1}{2}; u^* = \frac{1}{4} \longrightarrow x^* = 1.5; y^* = 1.5$$

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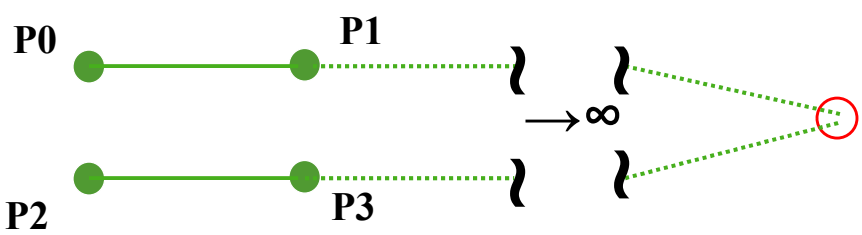
What if the Lines are Parallel ?

8



Well, technically they *do* intersect, but at infinity. This shows up in the math by the expression for t* and u* becoming infinitely large, that is, there would be a divide by zero.

So, you need to check the denominator before doing the division.

Since t* and u* would both be infinity in this case, which is most certainly **> 1**, that means that the line segments do not intersect.

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Linear Blending Shows Up in a Lot of Computer Graphics-Related Applications

9

You can linearly blend *any* two quantities with:

$$Q = (1-t)Q_0 + tQ_1$$

color, shape, location, angle, scale factors, •••

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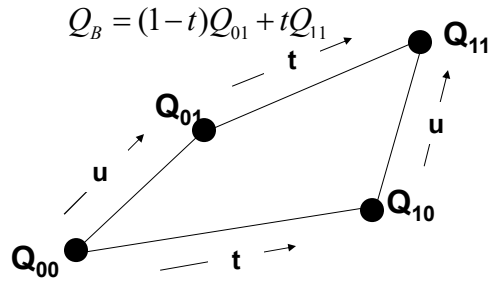
Bilinear Interpolation in a Quadrilateral

10

You can ***bilinearly*** blend *any* 2D quantities within a quadrilateral by writing the line blending equation twice and then blending the two lines with:

$$Q_B = (1-t)Q_{01} + tQ_{11}$$



$$Q_A = (1-t)Q_{00} + tQ_{10}$$

$$\boxed{Q_{AB} = (1-u)Q_A + uQ_B = (1-t)(1-u)Q_{00} + t(1-u)Q_{10} + (1-t)uQ_{01} + tuQ_{11}}$$

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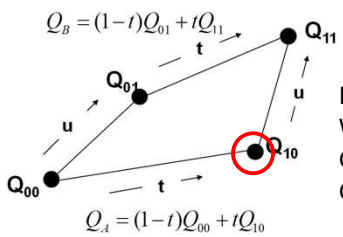
Bilinear Interpolation in a Quadrilateral

11

You can derive this equation by doing a blend-of-a-blend, like was shown on the last slide, or you can just look at how the terms have to become 0. or 1. whenever t = 0. or 1. and u = 0. or 1.



For example, right here t = 1. and u = 0. We know that at this location, the coefficient of Q_{10} has to be 1. and the coefficients of Q_{00}, Q_{01}, and Q_{11} have to all be 0.

$$Q_{AB} = (1-u)Q_A + uQ_B = (1-t)(1-u)Q_{00} + t(1-u)Q_{10} + (1-t)uQ_{01} + tuQ_{11}$$

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So, you can actually just write the bilinear equation "by inspection".

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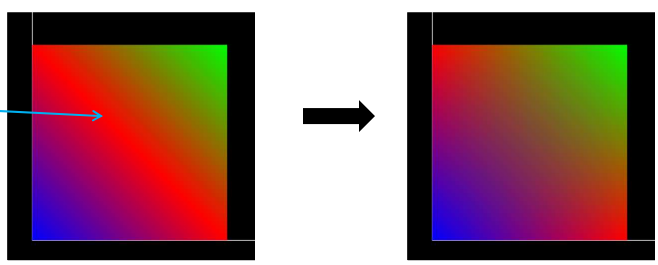
Bilinear Interpolation in a Quadrilateral

12

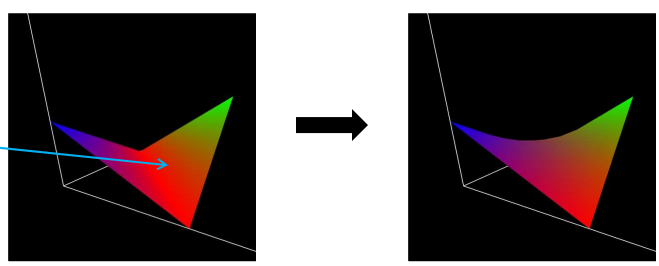
Drawing a quadrilateral as two triangles:

Drawing a quadrilateral with bilinear interpolation:

This diagonal should have some blue and green in it, but it doesn't



This should be a continuous surface, without crease



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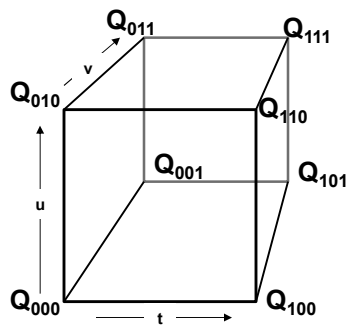
12

Trilinear Interpolation in a Hexahedron

13



$$Q = (1-t)(1-u)(1-v)Q_{000} + t(1-u)(1-v)Q_{100} + (1-t)u(1-v)Q_{010} + tu(1-v)Q_{110} + (1-t)(1-u)vQ_{001} + t(1-u)vQ_{101} + (1-t)uvQ_{011} + tuvQ_{111}$$

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Note: this works best with sides that are three pairs of quadrilaterals.

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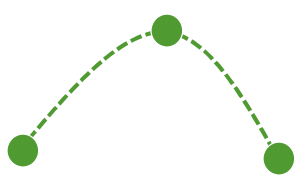
13

Higher-Order Interpolation

14

While linear interpolation is the most common type used in graphics and geometric applications, there is no reason we cannot use higher-order interpolants.

Second Order:



$$P(t) = A + Bt + Ct^2$$

$$A = P_0$$

$$B = -3P_0 + 4P_1 - P_2$$

$$C = 2P_0 - 4P_1 + 2P_2$$

Third Order: $P(t) = A + Bt + Ct^2 + Dt^3$

(We are going to discuss this more when we get to keyframe animation)

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14